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DOWNWARD FLOW OF A NONISOTHERMAL THIN LIQUID FILM WITH VARIABLE VISCOSITY

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Equations of a thin-film flow with linear dependences of viscosity and surface tension of the liquid on temperature are derived. The impact of various factors on the shape of the free boundary of the film are numerically analyzed.

Introduction. It is well known that, as a thin liquid film flows over the surface of an inclined flat substrate, local heating gives rise to complex structures, in particular, to a wave on the liquid surface, or liquid roller, which a region with substantial local thickening of the film near the heater [1, 2]. The physical reasons for the appearance of the wave were studied by Kabov et al. [3]; in the same work, a criterion for wave emergence was proposed.

Intense thermal loads may give rise to considerable temperature gradients. In this case, viscosity substantially differs from its mean value. The impact of viscosity on the structures induced on the film surface by thermocapillary forces is examined in the present work. The dynamic viscosity μ and the surface tension σ are assumed to linearly depend on temperature: $\mu(T) = \mu_0 - \mu_T(T - T_0)$, where $\mu_0, \mu_T = \text{const} > 0$, and $\sigma(T) = \sigma_0 - \sigma_T(T - T_0)$, where $\sigma_0, \sigma_T = \text{const} > 0$. The linear approximation for the $\mu(T)$ dependence is used from the following considerations. In the thin-layer approximation, as a rule, simplified equations of the transverse flow can be exactly integrated, and the problem reduces to solving one equation for film thickness. If viscosity is a variable quantity, then, generally speaking, no exact integration of governing equations is possible. This integration, however, becomes possible if viscosity depends on temperature in a linear manner; moreover, in this case, it also becomes possible to obtain relations, needed for the problem to be closed, between the film thickness, mass-flow rate of the liquid, and temperature variation in the case of stratified flows. It should be noted that, within the temperature range typical of the case under consideration, the relation between viscosity and temperature may always be fitted with a linear function. As an example, Fig. 1 shows the $\mu(T)$ dependence for 10 and 25% solutions of ethanol in water. If the temperature of the process ranges, for instance, between 20 and 40°C, then, we may use the straight-line segment shown in Fig. 1 as a fitting function. To approximate the viscosity versus temperature dependence in different temperature ranges, other linear functions may be invoked. We assume the thermal conductivity to be constant; this may be done based on the following considerations. According to [4], the measurement procedure for thermal conductivity includes the determination of the so-called acceptable temperature range, where thermal conductivity may be considered as a temperature-independent quantity. For liquids of interest (aqueous solutions of alcohols), the acceptable temperature range may be as wide as 10–20°C. At the same time, in such a temperature range, the dynamic viscosity varies appreciably (by tens of percents).

Formulation of the Problem. We consider a steady-state downward flow of a viscous incompressible liquid over an inclined flat surface (substrate) under the action of a gravity force. The substrate is inclined to the horizon at an angle α. We choose a Cartesian coordinate system (x, y) in which the y axis is orthogonal to the substrate. The liquid occupies the region $\Omega = \{(x, y)\colon -\infty < x < \infty,\ 0 < y < H(x)\}$, where H is the film thickness.

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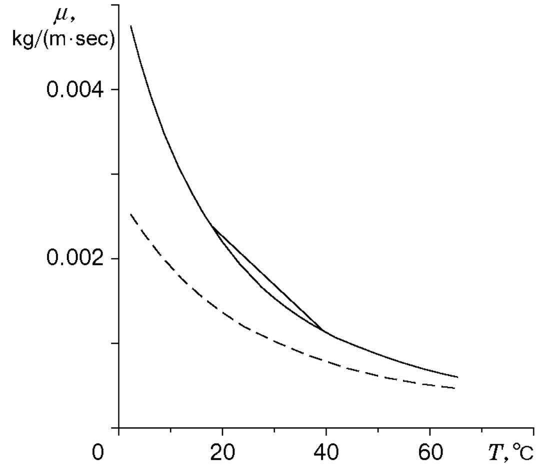

Fig. 1. Viscosity versus temperature: the dashed and solid curves refer to 10-% and 25-% solutions of ethanol in water.

If u and v are the velocity components, p is the hydrodynamic pressure, and T is the temperature of the liquid, then the film flow obeys the system of equations

$$\rho(uu_x + vu_y) = -p_x + \rho g \sin\alpha + 2(\mu u_x)_x + (\mu(u_y + v_x))_y, \tag{1}$$

$$\rho(uv_x + vv_y) = -p_y - \rho g \cos\alpha + 2(\mu u_y)_y + (\mu(u_y + v_x))_x, \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

$$uT_x + vT_y = \chi(T_{xx} + T_{yy}) \tag{4}$$

with the boundary conditions ($y = 0$)

$$u = v = 0, \tag{5}$$

$$æT_y - b_1(T - T_1(x)) = -q \tag{6}$$

and conditions at the interface ($y = H$)

$$uH_x = v, \tag{7}$$

$$æT_y + b_2(T - T_2(x)) = 0, \tag{8}$$

$$p_0 - p = -\sigma H_{xx}/(1 + H_x^2)^{3/2} + 2\mu[v_y - H_x(u_y + v_x) + H_x^2 u_x]/(1 + H_x^2), \tag{9}$$

$$\mu[2H_x(v_y - u_x) + (1 - H_x^2)(v_x + u_y)]/(1 + H_x^2) - \sigma_T T_s = 0. \tag{10}$$

Here T_0 is the mean temperature of the liquid, $\chi = æ/(\rho c_p)$, $æ$ is the thermal conductivity, c_p is the specific heat, ρ is the density of the liquid, p_0 is the gas pressure, q is the preset heat flux, and g is the free-fall acceleration. In view of its universality, the third-kind condition (6) is used as a thermal condition for the problem of interest: as $b_1 \to \infty$, we obtain $T|_{y=0} = T_1(x)$, i.e., a problem with a preset substrate-temperature distribution; as $b_1 \to 0$, we deal either with a thermally insulated flow or with a preset heat flux supplied to it. Thus, within this approach, it is possible to consider particular thermal regimes often encountered in practice. However, the quantities T_1, b_1, and q in relation (6) also retain their physical meaning with relation (6) preset in its general form. Indeed, setting third-kind conditions is nothing but using a simplified solution of the thermal problem for the substrate in the case where a preset distribution of temperature T_1 is realized not on the substrate surface but at a certain distance from the liquid. In this case, the heat-transfer coefficient b_1 is determined by the thickness of the layer in which temperature is disturbed by the downward liquid flow. Condition (8) at the free surface has a similar meaning. It should be noted that relations (6) and (8) by no means describe all possible thermal situations at the boundaries of the liquid film.

Derivation of Thin-Film Equations. We preset the scales of the dimensional quantities, i.e., film thickness, velocity of the liquid, longitudinal length, and order of deviation of temperature from its mean value, in the form

$$H_0 = \Big(\frac{3\Gamma\mu_0}{\rho g \sin\alpha}\Big)^{1/3}, \quad U = \frac{\mu_0}{\rho H_0}, \quad l = \Big(\frac{\sigma_0 H_0^2}{\rho U^2}\Big)^{1/3}, \quad \Delta T = \sup_x |T_1(x) - T_2(x)| + \sup_x \frac{q(x)}{æ H_0}.$$

Here $\mu_0 = \mu(T_0)$ and Γ is the preset mass-flow rate of the liquid.

We consider the problem in the thin-layer approximation, i.e., with $\varepsilon = H_0/l \ll 1$. In problem (1)–(10), using the formulas $x = lx_1$, $y = H_0 y_1$, $u = Uu_1$, $v = UH_0 v_1/l$, $H = H_0 h$, $\mu = \mu_0 - \mu_T \Delta T\theta$, $p = p_0 + \sigma_0 H_0 p_1/l^2 + \rho g l x_1 \sin\alpha - \rho g H_0 y_1 \cos\alpha$, $q = æ\Delta T q_1/H_0$, $T = T_0 + \Delta T\theta$, $T_1 = T_0 + \Delta T\theta_1$, and $T_2 = T_0 + \Delta T\theta_2$, we pass to dimensionless variables x_1, y_1, u_1, v_1, p_1, q_1, h, and θ. Using these variables and ignoring the minor terms in the powers of ε, we rewrite Eqs. (1)–(10) as (from here on, we omit the subscript 1)

$$-p_x + [(1 - \lambda\theta)u_y]_y = 0, \tag{11}$$

$$p_y = 0, \tag{12}$$

$$u_x + v_y = 0, \tag{13}$$

$$\theta_{yy} = 0 \tag{14}$$

with the boundary conditions ($y = 0$)

$$u = v = 0, \tag{15}$$

$$\theta_y - \mathrm{Bi}_1(\theta - \theta_1) = -q \tag{16}$$

and the conditions at the interface ($y = h$)

$$uh_x = v, \tag{17}$$

$$\theta_y + \mathrm{Bi}_2(\theta - \theta_2) = 0, \tag{18}$$

$$-p = Ax - Ch + h_{xx}, \tag{19}$$

$$u_y = -\mathrm{Ma}\theta'|_{y=h}/(1 - \lambda\theta). \tag{20}$$

Here the dimensionless criteria of similarity A, C, Ma, Bi_1, Bi_2, and λ are given by the formulas

$$A = gH_0^2 \cos\alpha/(U^2 l), \qquad C = gH_0 \sin\alpha/U^2,$$

$$\mathrm{Ma} = \sigma_T \Delta T H_0^2/(\mu_0 U l), \quad \mathrm{Bi}_1 = b_1 H_0/æ, \quad \mathrm{Bi}_2 = b_2 H_0/æ, \quad \lambda = \mu_T \Delta T/\mu_0.$$

The numbers A and C determine the values of the hydrostatic (longitudinal and transverse) components of the pressure gradient, the Biot numbers Bi_1 and Bi_2 determine the rate of heat transfer between the film and the substrate and between the film and the ambient (gas) medium, and the modified Marangoni number (Ma) specifies the strength of thermocapillary forces on the surface of the nonuniformly heated liquid film. The number λ determines the effect of variable viscosity on the processes of interest. Generally speaking, the second term in the right side of condition (19) is an out-of-order one: for small ε, this term is much smaller than the first one. However, if the angle of inclination α is small, the orders of the two terms are comparable.

Let us reduce problem (11)–(20) to one equation for the function h. Integrating Eq. (14), we obtain $\theta = Ny + M$. From the boundary conditions (16) and (18), it follows that

$$N = \frac{\mathrm{Bi}_1\mathrm{Bi}_2(\theta_1 - \theta_2) - \mathrm{Bi}_2 q}{\mathrm{Bi}_1\mathrm{Bi}_2 h + \mathrm{Bi}_1 + \mathrm{Bi}_2}, \qquad M = \frac{\mathrm{Bi}_2\theta_2 + (\mathrm{Bi}_2 h + 1)(q + \mathrm{Bi}_1\theta_1)}{\mathrm{Bi}_1\mathrm{Bi}_2 h + \mathrm{Bi}_1 + \mathrm{Bi}_2}. \tag{21}$$

From Eq. (12) and the boundary condition (19), we find $p = p(x) = Ax - Ch + h_{xx}$. Doubly integrating (11) and taking into account the boundary conditions, we obtain

$$u = -\frac{p_x y}{\lambda N} + \frac{1}{(1 - \lambda M)\lambda N}\Big(\frac{p_x}{\lambda N}[\lambda N h - (1 - \lambda M)] + \mathrm{Ma}\theta'|_{y=h}\Big) \ln\Big(1 - \frac{\lambda N y}{1 - \lambda M}\Big).$$

From the continuity equation (13), it follows that

$$v = -\int_0^y u_x(x,t)\,dt. \tag{22}$$

Substituting (22) with $y = h$ into the kinematic condition (17), we obtain the equation

$$\varphi h^3(h''' - Ch' + A) + \gamma h^2 \mathrm{Ma}\theta'|_{y=h} = \Omega \tag{23}$$

(Ω is the dimensionless mass-flow rate). Here

$$\gamma(x) = \frac{h^2}{(\lambda N)^2}\Big[\frac{\lambda N}{h} + \Big(1 - \lambda M - \frac{\lambda N}{h}\Big)\ln\Big(1 - \frac{\lambda N}{h(1-\lambda M)}\Big)\Big]; \tag{24}$$

$$\varphi(x) = \frac{h}{2\lambda N} - \frac{(1 - \lambda N/h - \lambda M)h^3}{(\lambda N)^3}\Big[\frac{\lambda N}{h} + \Big(1 - \lambda M - \frac{\lambda N}{h}\Big)\ln\Big(1 - \frac{\lambda N}{h(1-\lambda M)}\Big)\Big]; \tag{25}$$

$$\theta|_{y=0} = \frac{\mathrm{Bi}_2\theta_2 + (\mathrm{Bi}_2 h + 1)(q + \mathrm{Bi}_1\theta_1)}{\mathrm{Bi}_1\mathrm{Bi}_2 h + \mathrm{Bi}_1 + \mathrm{Bi}_2}; \tag{26}$$

$$\theta|_{y=h} = \frac{[\mathrm{Bi}_1\mathrm{Bi}_2(\theta_1 - \theta_2) - \mathrm{Bi}_2 q]h + \mathrm{Bi}_2\theta_2 + (\mathrm{Bi}_2 h + 1)(q + \mathrm{Bi}_1\theta_1)}{\mathrm{Bi}_1\mathrm{Bi}_2 h + \mathrm{Bi}_1 + \mathrm{Bi}_2}. \tag{27}$$

If $\lambda N \to 0$, then $\gamma \to 1/(2(1-\lambda M))$ and $\varphi \to 1\big/(3(1-\lambda M))$. Therefore, in the case of constant viscosity ($\lambda = 0$), Eq. (23) acquires a well-known form [5]. With h found, we can calculate the velocity-field components u and v, and also the pressure and temperature distributions.

Let us pose boundary conditions for Eq. (23). In previous studies, for the case of constant viscosity it was adopted that all perturbations decay as we go farther from the place of the local heating, either upstream or downstream, and the film thickness approaches its mean value. This value can be found from the preset mass-flow rate using the solution of the problem for stratified flows of an incompressible liquid of constant viscosity down an inclined plane. We assume that there are limits

$$\lim_{x\to\pm\infty} T_1(x) = T_1^{\pm} = \text{const}, \qquad \lim_{x\to\pm\infty} T_2(x) = T_2^{\pm} = \text{const}, \qquad \lim_{x\to\pm\infty} q(x) = q^{\pm} = \text{const}.$$

Generally speaking, the following relations are possible: $T_1^+ \neq T_1^-$, $T_2^+ \neq T_2^-$, $T_1^{\pm} \neq T_2^{\pm}$, or $q^{\pm} \neq 0$. In the case of variable viscosity, we have $H^+ \neq H^-$, where $H^{\pm} = \lim\limits_{x\to\pm\infty} H(x)$ and the values of H^+ and H^- differ from H_0. Assuming that the flow gradually becomes stabilized and does not differ from a stratified one with distance from the origin, we find a relation between the steady-state film thickness H^-, the mass-flow rate Γ, and the limiting values T_1^-, T_2^-, and q^-.

We construct a solution of the initial system (1)–(4) in the form $u = u(y)$, $v \equiv 0$, $p = p(y)$, and $T = T(y)$. In (5)–(10), we assume that $H = H^-$, $q = q^-$, $T_1 = T_1^-$, and $T_2 = T_2^-$. Then, these boundary conditions acquire the form

$$\begin{aligned} &u = 0, \qquad æT_y - b_1(T - T_1^-) = -q^- \qquad \text{for} \quad y = 0, \\ &p = p_0, \qquad u_y = 0, \qquad æT_y + b_2(T - T_2^-) = 0 \qquad \text{for} \quad y = H^-. \end{aligned} \tag{28}$$

Problem (1)–(4), (28) can be solved in close analogy with the well-known case of constant viscosity. Using the constructed solution, we find the sought relation between the parameters H^- and Γ:

$$\Gamma = \frac{\rho g \sin\alpha (H^-)^3}{\mu_0(1-\lambda M)k}\Big\{\frac{1}{2} - \frac{1-k}{k^2}[(1-k)\ln(1-k) + k]\Big\}, \qquad k = \frac{\lambda N}{1 - \lambda M}. \tag{29}$$

In calculating the values of N and M in formulas (21), we are bound to set $T_1 = T_1^-$, $T_2 = T_2^-$, $q = q^-$, and $h = H^-/H_0$. If the viscosity of the liquid differs only little from a constant one, then $k \to 0$ if $\lambda \to 0$, and (29) transforms into the classical Nusselt formula

$$\Gamma = \rho g \sin\alpha (H^-)^3/(3\mu_0).$$

Obviously, moving downstream from the origin, we can obtain a relation between the parameters H^+ and Γ in a similar manner. The resultant formula coincides with Eq. (29) where the superscripts "+"are used. Thus, the boundary conditions for Eq. (23) are set in the form

$$\lim_{x\to-\infty} h(x) = h^- = H^-/H_0, \qquad \lim_{x\to\infty} h(x) = h^+ = H^+/H_0. \tag{30}$$

Numerical Algorithm. The computations were carried out according to the following iterative algorithm. First, using the given mass-flow rate Γ, the values of $H^\pm$ were determined from (29) by the method of interval bisection. Then, an approximate value of the function $h(x)$ was set by the formula

$$h^0(x) = \begin{cases} h^-, & x < -1, \\ h^- + (h^+ - h^-)[\sin(\pi x/2) + 1], & -1 < x < 1, \\ h^+, & x > 1. \end{cases}$$

Using this value, from formulas (21), (26), and (27), we found the initial values of temperatures $\theta|_{y=0}$ and $\theta|_{y=h}$ and, from formulas (24) and (25), the initial values of the functions γ and φ, denoted below as γ^0 and φ^0. Further, for Eq. (23), we constructed its linear analog

$$\varphi^0[(h^0)^3(h''' - Ch') + (h^0)^2hA] + \gamma^0 h^0 h \mathrm{Ma}(\theta^0)'|_{y=h} = \Omega. \tag{31}$$

Equation (31) with conditions (30) was solved numerically, and then the next iteration step was started: new values of $\theta|_{y=0}$, $\theta|_{y=h}$, etc., were found. The computational domain was chosen to be large enough for the perturbations to decay at its borders.

To numerically solve Eq. (31) with conditions (30), we used the following five-point difference scheme [6]:

$$\varphi_i^0\Big[(h_i^0)^3\Big(\frac{-h_{i-2} + 2h_{i-1} - 2h_{i+1} + h_{i+2}}{2\delta^3} - C\,\frac{h_{i+1} - h_{i-1}}{2\delta}\Big) + (h_i^0)^2 h_i A\Big]$$
$$+ \gamma_i^0 h_i^0 h_i \mathrm{Ma}((\theta^0)'|_{y=h})_i = \Omega \qquad (i = 3, \ldots, N-2). \tag{32}$$

To construct the numerical scheme, the solution interval $(-L, L)$ was divided into $N-1$ equal sub-intervals with a division step δ. The value of L here is large enough, and the flow at $x = -L$ and $x = L$ does not differ from a stratified one. For this reason, the following conditions are added to the system of linear equations (32):

$$h_1 = h_2 = h^-, \qquad h_{N-1} = h_N = h^+. \tag{33}$$

Conditions (33) imply that conditions (30) are satisfied, and $h' \to 0$ as $x \to \pm\infty$.

Computation Results. In the computations, the material constants of the liquid corresponded to a 25-% solution of ethanol at 20°C. The heat-transfer conditions were as follows: $T_1 = T_2 = 20$°C and $\mathrm{Bi}_1 = \mathrm{Bi}_2 = 0.15$. Such values of Bi_1 and Bi_2 correspond to weak heat transfer between the liquid and substrate and between the liquid and gas phase. This complies with the test conditions adopted in [1, 2]. An increase in Bi_1 and Bi_2 just results in weakening of thermocapillary and other thermal effects.

Two variants of setting the density q of the heat flux from the substrate to the liquid were considered.

Variant A. The liquid flow enters the heated substrate:

$$q(x) = \begin{cases} 0, & x < 0. \\ q_0, & x > 0; \end{cases}$$

Variant B. The liquid film is locally heated:

$$q(x) = \begin{cases} 0, & x < -b/2, \\ q_0, & -b/2 < x < b/2, \\ 0, & x > b/2. \end{cases}$$

Here b is the heated length of the substrate and $q_0 = \mathrm{const}$ is the heater intensity.

Figure 2a shows the shape of the free boundary of the liquid film for the thermal regime A. The heater position is marked with a heavy line on the abscissa axis. The computations were performed for $q_0 = 0.1$ W/cm^2 and $A = 0.337 < C = 1.68$ for the following three cases (the numbers at the curve in Fig. 2 refers to the corresponding variants): 1) $\sigma = \sigma(T)$ and $\mu = \mu(T)$; 2) $\sigma = \sigma(T)$ and $\mu = \mathrm{const}$; 3) $\sigma = \mathrm{const}$ and $\mu = \mu(T)$. Figure 2b shows the computed shape of the free surface for the thermal regime B. The computations were performed for $b = 0.2$ cm, $q_0 = 0.1$ W/cm^2, and $A = 0.337 < C = 1.68$. It should be noted that the heated sector on the substrate surface makes the film thickness locally increase near the front edge of the heater (liquid roller); subsequently, the film thickness decreases, attains a minimum and, as it further increases, approaches a constant value given by conditions (30). An analysis of the computation data gained shows that the height of the wave that arises on the film surface is governed by the Marangoni number, whereas the degree of film thinning depends on the parameter λ.

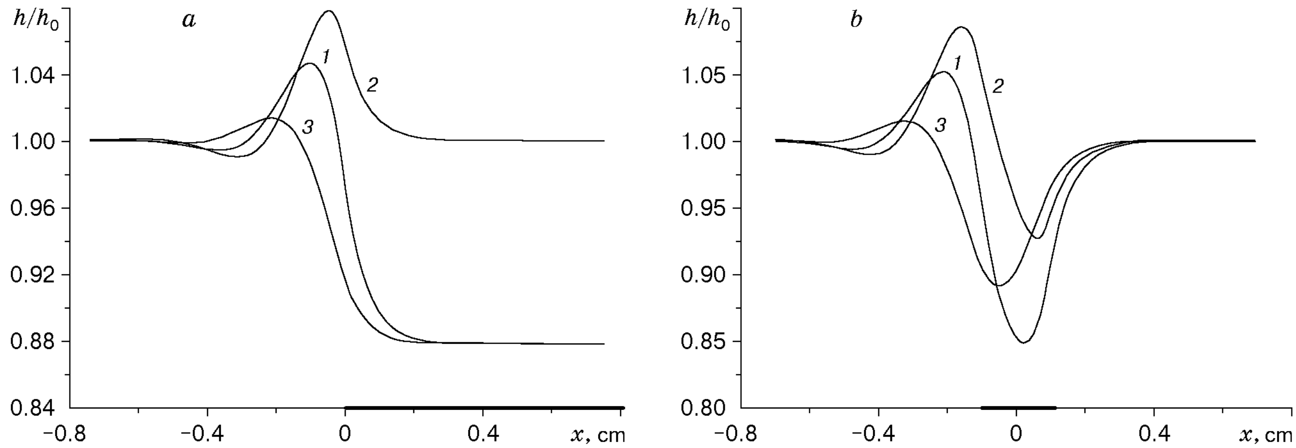

Fig. 2. Relative film thickness as a function of the longitudinal coordinate for a liquid flow that enters a heated region on the substrate (a) and for a liquid flow undergoing local heating (b): 1) $\lambda = 0.0941$ and Ma $= 0.118$; 2) $\lambda = 0$ and Ma $= 0.118$; 3) $\lambda = 0.0941$ and Ma $= 0$.

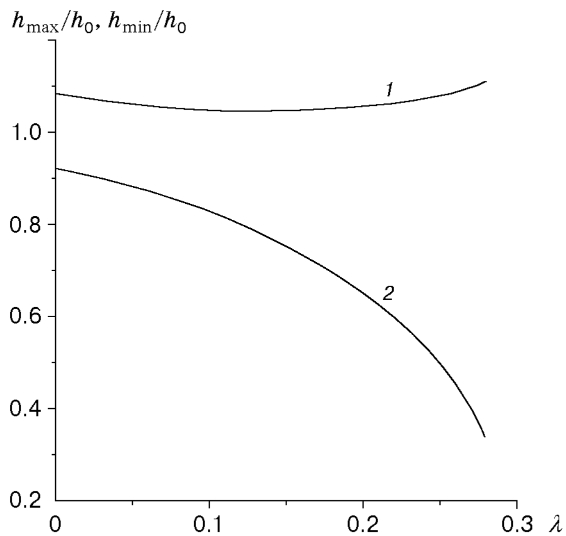

Fig. 3

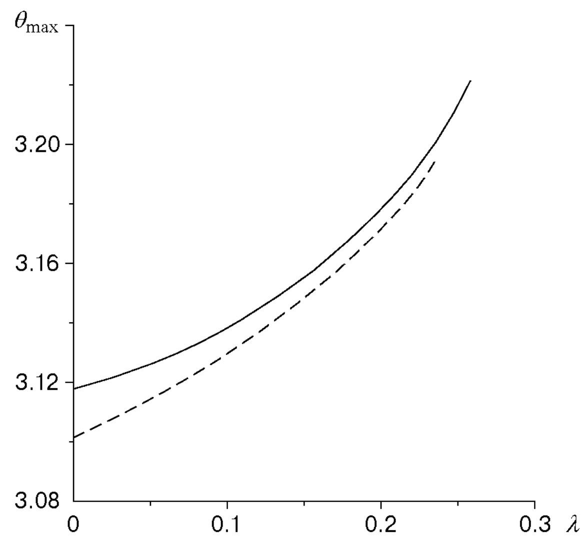

Fig. 4

Fig. 3. Maximum and minimum thicknesses of the film versus the parameter λ for the thermal regime B: 1) $h_{\max}/h_0$; 2) $h_{\min}/h_0$.

Fig. 4. Highest surface temperature versus the parameter λ: the dashed and solid curves refer to the variants A and B, respectively.

This conclusion is supported by the computation data plotted in Fig. 3. Note that the wave height is almost independent of λ, whereas the minimum film thickness rapidly decreases with increasing this parameter. Another characteristic of the film used in analyzing the possibility of its rupture is the highest surface temperature. The dependence $\theta_{\max}(\lambda)$ is shown in Fig. 4.

Thus, in studying the dynamics of nonisothermal film flows, one should take into account the temperature dependence of viscosity, since this dependence determines the most important characteristics of the liquid film: the smallest thickness of the film and the degree of heating of its free surface.

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